Competitive screening and information transmission

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Abstract

We consider a model in which schools and colleges compete for high-ability students, which are independently identified through a costly screening procedure. This independence creates a channel through which students' preferences affect the strategic interaction between schools: students with competing offers accept the most preferred one, increasing the screening costs of unpopular schools. When preferences between schools are more heterogeneous, schools screen more, increasing the proportion of students with multiple offers, but paradoxically reducing the extent to which their preferences determine their outcomes. By observing the students' schools of origin, colleges can free-ride of the fierce competition that occurs during screening.

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1 Introduction

Colleges, and in many cases schools, often receive more applications than they can accept. At the same time, in some countries and systems, these institutions value the composition of the admitted cohort in terms of their abilities in relevant fields. In other words, it is often the case that not only colleges but also schools want to admit the "best and the brightest" among their applicants.

In places where education is provided only by the public sector, this is typically not the case: students are often admitted based solely on their locations. Under the global trend of marketization of education systems [Walsh, 1995, Spring, 1998], there are many other circumstances in which a competitive procedure is used. In Hong Kong, for example, admission to prestigious private schools or even nurseries [Cheung, 2015] is based on interviews and skill-based criteria that are determined independently by each institution. This is done through every step of the education system, from kindergarten to college. Similar procedures are used in Brazil, China, and Singapore [McCowan, 2004, Yan, 2017, Chan and Tan, 2008].

To identify those high-ability students from the pool of applicants, however, involves engaging in costly examinations, interviews, and other screening methods. As students move through the education system, the institutions the student has attended can be used as a signal of their ability. For example, if high schools also screen applicants, the fact that an applicant graduated from a very competitive high school may signal to colleges that the student may have strong ability.

This paper evaluates the effect of these screening strategies on the matching of students to institutions. We take into account three aspects of this process. The first is that screening decisions affect other institutions' outcomes, because the same student may be identified as high ability by more than one institution. The second is that students' preferences affect the returns from screening, as a student will only accept an offer from a less preferred institution if she did not receive an offer from a more preferred one. The third is how the information generated by this knowledge about screening and preferences is used by institutions at the next level in the educational sequence.

By incorporating these elements into a simple model, we provide insights into multiple aspects of the process in closed-form solutions. In our model, there are two schools and two colleges. Students can apply to the schools or colleges at no cost. This abstracts the strategic aspect of where to apply and, as a result, isolates the effect of the intersection of schools' offers on outcomes and schools' incentives. When a school engages in screening and identifies a high-ability student, the other school may also identify that same student as such. The student will then have to choose between the offers made based on her preference over these schools. Therefore, the less popular school will see a greater proportion of its offers rejected than the more popular school. That translates into a higher cost per high-ability student admitted, or, equivalently, a lower marginal gain from screening for the former.

These different marginal returns from screening result in a surprising relation between screening costs and the schools' success in obtaining high-ability students. We show that while in general a reduction in screening cost improves outcomes for both schools, there is a point at which further reductions in cost have opposite effects on the student cohorts obtained by them: the popular school accepts more high-ability students, while the less popular one obtains less. The reason behind this is that lower screening costs increase the incentive for schools to engage in screening and, as a result, it also increases the number of students with offers from both popular and unpopular schools. As unpopular schools are negatively affected by an increase in the number of offers that students have, when costs are low enough that effect becomes stronger than the reduction in the marginal cost of screening (Proposition 4).

Another interesting observation is how changes in students' preferences affect schools' strategies and, as a result, their own welfare. As students' preferences become less correlated (that is, the less popular school increases in popularity), this creates two opposite effects in both schools. The return from screening for the popular school decreases and the return from screening for the less popular school increases, due to the effect created by simultaneous offers. We show, however, that the former effect is proportional to the amount of screening conducted by the more popular school, whereas the latter is proportional to the amount of screening conducted by the less popular school. As a result, there is an overall increase in the equilibrium amount of screening performed by the schools, which also leads to an increase in the number of students who receive multiple offers (Proposition 5). While this raises the total number of high-ability students matched to the schools, it paradoxically reduces the number of those who are matched to their most preferred school (Proposition 6). This is because when a student changes her preference toward the less popular school, she will prefer to be matched to a school that screens fewer students. As a result, the likelihood that she will receive an offer from her top school is lower.

Our model also explores the transmission of information that takes place when colleges make screening decisions based on which schools the students come from. The principle is simple: colleges are aware that the proportion of high-ability students is higher at the most popular schools in equilibrium, and can explore that fact when screening applicants. Moreover, that ability to consider this information should also depend on students' preferences over colleges: if students have common preferences over colleges, the most preferred college can make its decisions independently (because all its offers will be accepted). Therefore, whatever informational advantage there is to be exploited will also be fully available to the top college. We show, however, that this does not translate into an advantage for the top college: the number of high-ability students obtained by both colleges depends on the total number of those in the schools, but not on how they are distributed among them. The colleges do enjoy the benefits that are generated by the schools' reaction to changes in preferences, though: colleges obtain better cohorts, in equilibrium, when students' preferences over schools are less homogeneous. In these cases, colleges "free ride" on the increased screening performed by the schools (Proposition 7).

We also explore the sources of inefficiency that emerge from this decentralized competitive screening process, and suggest policy interventions that could improve upon equilibrium outcomes. That analysis points to methods such as partitioning the students to be screened between schools and asking students to reveal their preferences to improve their allocations. It also shows, however, that they involve trade-offs: partitioning the students reduces redundant screening but increases marginal costs, and requiring students to reveal their true preferences limits efficiency. We also explore the possibility of using application costs combined with screening as a method for preventing low-ability students from applying to schools.

1.1 Related Literature

Our article is closely related to recent articles that focus on information acquisition in matching problems. There is a line of research that analyzes the conditions under which assortative matchings are produced. While previous work showed that search costs in the form of time discounting may lead to non-assortative matchings in a market with transfers, Atakan [2006] showed that this is not the case if the search cost is constant, because the cost imposed by time discounting is heterogeneous among agents. In a setup with some similarities to ours, Lien [2006] evaluates the role of a limited number of interviews in college admission outcomes. In his model, colleges choose which students to interview based on noisy public signals about students' match quality. Depending on how informative the public signals are, high-ability students may "fall through the cracks," due to the fact that lower-ranked colleges shy away from interviewing high-ranked students to avoid "wasting" interviews. In a related article, Kadam [2014] considers a model in which firms engage in costly interviews to learn the value of a binary "fitness" of students. In equilibrium, firms spread their interviews among "star" candidates, medium-ranked students, and "safe bets."

Lee and Schwarz [2017] also consider firms' (or colleges') screening decisions. In their model, firms can hire only one worker and have different valuations for them. Unlike our model, firms can coordinate on which workers each firm will interview. This ability to coordinate reduces the competitive aspect that the overlap has on firms' expected gain from screening. In these cases, however, the identity of the firm that overlaps with each other also matters: when two firms interview the same set of workers, a rejection from a worker implies reduced competition for the other workers. Unlike our model, however, workers' preferences are uniformly drawn and therefore there is no role for that in firms' screening strategies.

In a model in which workers and firms share the surplus generated by their matches, Josephson and Shapiro [2016] show that costly screening by firms may prevent efficient matching because potentially strong candidates are not interviewed to avoid competition from more productive firms. Other articles, such as Che and Koh [2016] and Hafalir et al. [2018], also consider the role of competition between colleges when selecting students.

Ely and Siegel [2013] evaluate how revealing interview decisions by other firms affects equilibrium outcomes. They show that when firms can observe other firm's interviewing decisions, that information can be better exploited by the most preferred firm. This happens because workers who face multiple offers accept the one from the most preferred firm. This makes the choice of interviewing by less preferred firms very informative, leading the top firm to interview as well. In our model, although firms can observe each other's aggregate screening decisions, this is not the case for individual students. The choice of how many students to screen, however, does have a similar effect on the less desirable school or college, which must anticipate the fact that the more the other school screens, the more likely it will be that they will send offers to the same student.

Chade et al. [2014] consider the effects that application costs have on students and colleges' behavior in equilibrium. Given these costs, students face a portfolio choice problem in their application decision. In their model, both students and colleges act strategically and make decisions under uncertainty. In equilibrium, there may not be assortative matching, because weaker students may apply more aggressively, while smaller but weaker colleges may impose higher standards.

One important characteristic that distinguishes our model from those above is that colleges use students' school of origin as an endogenous signal of the applicants' ability. Arrow [1973] introduced the concept of higher education as a "filter," where the screening performed by colleges is used by firms as a signal of the student's ability. His baseline model, like ours, makes the simplifying assumption that education does not change the ability of a student, but instead that the screening process produces a signal about that student's unobservable ability.

In a dynamic model in which the quality of past cohorts is partially used to produce school rankings, Herresthal [2020] shows that the informativeness of the rankings are enhanced in a steady-state equilibrium if a greater proportion of students are selected based on merit and when the costs of attending non-local schools are reduced. MacLeod and Urquiola [2015] also consider the case where a college's reputation is based on the quality of the students that it selects. Interestingly, colleges' concern for reputation is endogenous: while their objective is to maximize enrollment, the ability to produce good job market signals for their students in a competitive environment incentivizes being more selective. Similarly, while there are no peer effects between students in their model the fact that cohorts' expected ability is one of the factors used by firms when deciding on wages for new employees causes students to prefer colleges with high-ability colleagues. Conley and Önder [2014] empirically evaluate how much information on the productivity of economists is obtained by observing the ranking of the department where he or she obtained his or her PhD. Perhaps surprisingly, they find that the ranking of a student in a program is often a better predictor of future performance than the ranking of the department itself. Nevertheless, Baghestanian and Popov [2014] show that the publication market values the signals from the Alma mater of — and the position held by — the author.

Still, regarding the use of the signaling value of the school of origin, one aspect that is not explored in our article is how schools may strategically inflate students' grades, with the objective of placing them in better colleges or jobs. Chan et al. [2007] present a model in which schools inflate grades to oversell their students in equilibrium. This grade inflation, however, reduces matching efficiency and in fact harms schools. Popov and Bernhardt [2013] present empirical evidence of grade inflation and similar theoretical results.

The remainder of the paper proceeds as follows. Section 2 introduces the model, the equilibrium choices for the schools and for the colleges, and the impact of assuming that students act strategically. In section 3 we analyze the effects of students' preferences and screening cost on outcomes and welfare. In section 4 we discuss the sources of inefficiency in equilibrium and evaluate the feasibility and efficacy of different policy interventions to tackle them. In section 5 we discuss the impact that some of the assumptions that we make have in our main results. Section 6 concludes. All proofs are relegated to the appendix.

2 Model

Our education system consists of two schools, S_1, S_2, and two colleges, C_A, C_B. Schools S_1 and S_2 are entry-level schools — that is, there is no pre-requisite for attending them, whereas in order to be accepted at colleges C_A or C_B, students must have attended a school. Institutions have their capacities: schools S_1 and S_2 can accept masses of at most $q > 0$ students, and colleges C_A and C_B can accept at most $Q > 0$ students each.

A share σ of students prefer school S_1 over school S_2, where $0 \leqslant \sigma < \frac{1}{2}$. That is, the majority of students prefer school S_2 over S_1. All students prefer college C_A over C_B.

Students' characteristics are multidimensional. They have quantifiable skill levels in each dimension: mathematical, artistic, linguistic, and etc. Students whose skills are within a subset of this space are deemed to be *high-ability* by schools and colleges. Students within another, disjoint, subset of that space, are deemed *very undesirable*. The remaining students, who are neither high-ability nor very undesirable, are considered *acceptable*. Whenever the distinction is not important, we refer to students as being high or low-ability—where the latter combines very undesirable and acceptable students. While schools agree on what constitutes a very undesirable skill set, colleges might deem some students who are acceptable to schools as very undesirable. We assume that students do not know their own skill levels, and that a student' skill levels are statistically independent of their preferences between schools.

Students can be acquired from two sources. One source is an *"external" pool* with a continuum of students, with total mass normalized to 1, where a fraction $\alpha > 0$ of them are high-ability, and a non-zero fraction of them are very undesirable students.[1] Institutions cannot directly observe the ability of students but must independently conduct a costly but perfect screening process to identify those who are high ability. The other two are *"internal" pools* with an arbitrarily large continuum of students for the two levels of education. All of the students from the internal pools are acceptable. None of them are, therefore, deemed as high ability or very undesirable. Each individual student has zero mass.

One way to interpret the internal pools is to think of them as students who come from earlier stages of the institution itself (for example, students from a college's high school), or those who have a diploma from the standard education system, while those from the external pool are international students, for which there is more uncertainty about the quality of their education, but also contains some extremely talented students.

We assume that although schools may have an impact on a student's abilities, this impact

[1] This normalization is not consequential to our results. As long as the total mass of students and the fraction of high-ability students are large enough, all of our results will hold through.

is uniform across schools and cohorts. As a result, students who are deemed to be high ability by schools are also the ones who are deemed as such by colleges.

2.1 Admission

We consider a two-stage admission process, first by schools and then by colleges. In the first stage, students can make costless applications to schools S_1 and/or S_2. Given the external pool of applicants, schools independently choose the mass of applicants to screen. After screening, each school knows the abilities of their screened students. Then schools send offers to students, and students decide whether to accept an offer among those received, if any, or remain unmatched. Schools may fill the remaining seats with students from the internal pool.

In the second stage, after the matching process from the schools ends, students apply to colleges, which perform the same kind of screening that schools performed for the external pool. In the case of colleges, however, the screening is made for the set of students who attended schools S_1 and S_2. Colleges make independent screening decisions for the students who come from each one of the two schools. Similar to the first stage, after screening, each college knows the abilities of their screened students. Then, colleges send offers to students and students decide whether to accept an offer or remain unmatched. Colleges may also fill the remaining seats with students from the internal pool.

The assumption that students are divided into two "pools" with these characteristics has the effect of making the model tractable and narrows the qualitative results to those related to our main research questions — namely, the interaction between students' behavior and schools' competitive screening, and the transmission of information from schools' equilibrium cohorts to colleges' screening choices.

To observe this, notice that if schools only sent offers to high-ability students, then all cohorts would consist only of high-ability students and there would be no meaningful questions to consider about the transmission of information. Moreover, it is not plausible to assume that schools are willing to leave many seats empty simply because they could not acquire enough high-ability students to fill them. Therefore, our model must incorporate a method for schools to fill their remaining seats without resorting to screening. This is the methodological role of the internal pools.

For this and the next section of the paper, we will assume that students will always apply to both schools, and when facing a choice between offers from both schools, they always accept the offer from the most preferred school. In principle, a student could accept an offer from a less desirable school, if that increases her likelihood of being accepted at a more preferred college later on. We will relax this assumption in Section 2.5 and show that any other choice is dominated. Therefore, our analysis does not rely on naïve or non-strategic student behavior.

2.2 Screening

Consider the problem that a school or college has when screening a pool with a mass η of applications, of which a mass P_h of them are high-ability students. The screening procedure consists of using increasingly costly methods to identify high-ability students from the pool of applicants. For example, a school first selects students who have a maximum GPA. Students are sampled from the pool until one who satisfies this (inexpensive) criterion is found. The probability that a student sampled in this process has high ability is therefore $\frac{P_h}{\eta}$. Next, a more costly procedure is used to identify more high-ability students. One example involves evaluating recommendation letters from students with high (but not the maximum) GPAs. This more costly procedure is now used in a pool that has a proportion $\frac{P_h - x}{\eta}$ of high-ability students, where x is the mass of those identified using the previous, less expensive method. This notion can, therefore, be extended to a continuum of increasingly costly methods. We assume that the costs of the different screening methods are bounded from below by zero and from above by some value that is large enough so that no school or college in our model would get to the point of using that method. Moreover, we assume that the density of screening methods cost is uniform. In other words, the mass of screening methods available in the neighborhood of a given cost is not different from the mass in the neighborhood of another cost. Therefore, the cost associated with the method used to identify high-ability students after x methods were previously used is κx, for some $\kappa > 0$.

The marginal cost of screening consists of two parts: one that follows the cost of the method used in each step, and one that is proportional to how unlikely it is that a sampled student has high ability. The cost that a school incurs in identifying a mass λ of high-ability students from a pool of η applicants that has a mass P_h of high-ability students is therefore the following:

$$\mathcal{C}\left(\kappa; \eta, P_h, \kappa\right) = \int_0^\lambda \kappa x \frac{\eta}{P_h - x} dx.$$

In the expression above, $\frac{\eta}{P_h - x}$ is the inverse of the probability that a student sampled and evaluated with that method has high ability. The resulting cost function has two important characteristics. First, it results in tractable closed-form solutions for equilibrium strategies and cohort compositions. As we will argue in section 5, this is especially important if we want to consider students' strategic behavior. Secondly, it satisfies two key properties that the screening cost function should have: (i) screening costs are convex on λ, and (ii) the marginal costs of screening decrease with the proportion of high-ability students in the pool. Item (i) is important to prevent returns from screening that lead to corner solutions (i.e. screen all students or screen none of them,) while (ii) guarantees that the information generated by the schools' screening process, in the form of different proportions of high-ability students in their cohorts, is potentially valuable to colleges.

Institutions cannot directly observe the ability of students but must independently conduct a costly but perfect screening to identify their abilities.

In the school admission, if each school S_i sends offers to a mass λ_i of high-ability students from the external pool, the (expected) mass of those who will accept the offer are, for each school, the following:

$$H_1 = \lambda_1 - (1-\sigma)\frac{\lambda_1\lambda_2}{\alpha} \text{ and } H_2 = \lambda_1 - \sigma\frac{\lambda_1\lambda_2}{\alpha}$$

While students who receive only one offer will accept it, students who receive more than one offer will choose which one to accept based on their preference. Given the values of λ_1 and λ_2 and the fact that every high-ability student has the same probability of being independently identified as such by a given school, the mass of students who receive offers from both schools is $\frac{\lambda_1\lambda_2}{\alpha}$. When students follow their preferences, a proportion σ of those students who receive offers from both schools will accept school S_1, and the remaining students will accept school S_2.

In college admission, if college C_i sends offers to λ_i^j high-ability students from school S_j, which has H_i high-ability students, students who receive offers from both colleges will choose to go to C_A, as every student prefers college C_A over C_B. If there is enough capacity in each college so that these will not be binding,[2] the two masses of high-ability students that the colleges acquire will be:

$$H_A = \lambda_A^1 + \lambda_A^2 \quad \text{and} \quad H_B = \lambda_B^1 - \frac{\lambda_B^1\lambda_A^1}{H_1} + \lambda_B^2 - \frac{\lambda_B^2\lambda_A^2}{H_2}.$$

For clarity, we will, from this point on, refer to the choice of λ that schools and colleges make as their "screening" choices. Implicitly, however, when we say that a school "screens λ

[2]We will, in fact, make this assumption explicitly later on.

students," we mean to say that the school will screen until λ students are identified as high ability.

Each high-ability student from the pool of applicants is equally likely to be identified, and this identification is independent between schools. The screening policy for a school or college consists, therefore, of choosing an expected mass of high-ability students from the external pool to select by screening, λ. After both schools or colleges perform that screening process simultaneously, they will proceed to send offers to students (also simultaneously). Students who do not receive any offers will remain unmatched. Students who receive offers from one or both schools may accept at most one of them.

2.3 Competition

Schools and colleges have lexicographic preferences over their cohorts: first, they prefer more students than less, and then have linear preferences over the proportion of high-ability students. Since schools can always fill seats with unscreened students from the internal pool, however, this will never lead to schools having to choose between more students or better ones. Very undesirable students, moreover, are unacceptable for schools and colleges. Therefore, they only admit unscreened students from internal pools.[3] The result of these assumptions is that schools will always select as many high-ability students as possible, subject to cost and strategic considerations, and then fill the remaining seats with students from the internal pool.

In the first stage admission, when school S_i decides to screen λ_i high-ability students, its payoff is

$$U_1(\lambda_1, \lambda_2) = H_1 - \mathcal{C}(\lambda_1; 1, \alpha, \kappa) = \left(\lambda_i - (1-\sigma)\frac{\lambda_1\lambda_2}{\alpha}\right) - \kappa\int_0^{\lambda_i}\frac{x}{\alpha - x}dx, \text{ and}$$
$$U_2(\lambda_1, \lambda_2) = H_2 - \mathcal{C}(\lambda_2; 1, \alpha, \kappa) = \left(\lambda_2 - \sigma\frac{\lambda_1\lambda_2}{\alpha}\right) - \kappa\int_0^{\lambda_2}\frac{x}{\alpha - x}dx.$$

In the second stage admission, when college C_i decides to screen λ_i^j high-ability students from each school S_j, which has q students, H_j of which are of high abilities, college C_i's

[3] Notice that, even though schools do not admit "very undesirable" students, some of them might be deemed "very undesirable" by colleges. Therefore, colleges do not admit unscreened students from schools as well.

payoff is $U_i(\lambda_A, \lambda_B; q, H_1, H_2) = H_i - \mathcal{C}(\lambda_i^1; q, H_1, \kappa) - \mathcal{C}(\lambda_i^2; q, H_2, \kappa)$ so that

$$U_A(\lambda_A, \lambda_B; q, H_1, H_2) = \lambda_A^1 + \lambda_A^2 - \kappa \left(\int_0^{\lambda_A^2} \frac{qx}{H_2 - x} dx + \int_0^{\lambda_A^1} \frac{qx}{H_1 - x} dx \right), \text{ and}$$

$$U_B(\lambda_A, \lambda_B; q, H_1, H_2) = \lambda_B^1 - \frac{\lambda_B^1 \lambda_A^1}{H_1} + \lambda_B^2 - \frac{\lambda_B^2 \lambda_A^2}{H_2} - \kappa \left(\int_0^{\lambda_B^2} \frac{qx}{H_2 - x} dx + \int_0^{\lambda_B^1} \frac{qx}{H_1 - x} dx \right).$$

Notice that schools and colleges are not profit maximizers in our model. There are multiple reasons why we believe that this is a sensible choice. First of all, schools are often inherently interested in their reputations. This is not unusual for universities, or schools funded by private foundations, for example. Another reason concerns cases in which schools' funding is public but conditional on satisfying good outcomes, such as exams and other skill-based evaluations. Selecting high-ability students, therefore, maximizes the probability of continued funding. Finally, we could also consider a situation in which schools are profit-maximizers, where their ability to charge higher tuition depends on how their students perform in standardized tests (since these are used as signals by parents), but that they can only change tuition fees for the following year. As a result, they will want to fill their seats with as many high-ability students as possible.

2.4 Equilibrium

In this section, we assume that students are not strategic and simply accept the offer from the most preferred school or college, if any. By doing so, we model the strategic interactions as a two-stage complete information game between schools and colleges. This greatly simplifies our analysis for the remainder of this section and section 3. In subsection 2.5, however, we relax this assumption and evaluate students' incentives. This analysis will show that this truthful behavior is indeed supported by the fact that deviations from this truthful behavior are dominated by it.

Our education system is a two-stage complete-information game where the school admission game (first stage) is a competition between schools S_1 and S_2, and the college admission game (second stage) is a competition between colleges C_A and C_B. All schools and colleges are risk-neutral. We employ subgame-perfect Nash equilibrium as our solution concept. Notice that, despite the fact that colleges play after schools, schools' payoffs are fully determined by their actions in their stage-game. Therefore, we can solve the entire game by solving first for the schools' equilibrium, and then the colleges'.

In the school admission stage, schools S_i independently choose $\lambda_i \in [0, q]$, which are the masses of high-ability students from the external pool to be screened. The payoff of school S_i is $U_i(\lambda_1, \lambda_2)$ as defined above. A pair of schools' screening choices $\lambda_S^* = (\lambda_1^*, \lambda_2^*)$ is a (Nash) equilibrium if $U_1(\lambda_S^*) \geqslant U_1(\lambda_1, \lambda_2^*)$ and $U_2(\lambda_S^*) \geqslant U_2(\lambda_1^*, \lambda_2)$ for all λ_1 and λ_2.

We will assume that the value of q (the capacity of schools S_1 and S_2) is large enough such that no school will, in equilibrium, have only high-ability students. More specifically, we have:

Assumption 1. *The capacity of schools are such that $q \geqslant \alpha$.*

Under this assumption, the equilibrium of the first-stage subgame is as follows:

Proposition 1. *Under Assumption 1, there is a unique Nash equilibrium of the school admissions game, where the equilibrium screening and the masses of high-ability students in schools S_1 and S_2 are*

$$\lambda_1^* = \frac{\kappa(\kappa+2) + \sigma(\sigma+1) - A}{2\sigma(\kappa+\sigma)}\alpha, \ \lambda_2^* = \frac{\kappa(\kappa+2) + (1-\sigma)(2-\sigma) - A}{2(1-\sigma)(\kappa+1-\sigma)}\alpha,$$

$$H_1^* = \frac{\alpha\left(A + \sigma(1-\sigma) - \kappa^2\right)\left(\kappa(\kappa+2) + \sigma(\sigma+1) - A\right)}{4\sigma(\kappa+\sigma)(\kappa+1-\sigma)}, \text{ and}$$

$$H_2^* = \frac{\alpha\left(A + \sigma(1-\sigma) - \kappa^2\right)\left(\kappa(\kappa+2) + (1-\sigma)(2-\sigma) - A\right)}{4(1-\sigma)(\kappa+\sigma)(\kappa+1-\sigma)}$$

where $A = \sqrt{\kappa^4 + 4\kappa^3 + 2\kappa^2(2-\sigma(1-\sigma)) + 4\kappa\sigma(1-\sigma) + \sigma^2(1-\sigma)^2}$.

In the college admissions game, given that colleges know the game played by the schools in the first stage, they can infer that there are $H_i < q$ high-ability students in school S_i, and that there are q students in each school. College C_i independently chooses masses $\lambda_i = (\lambda_i^1, \lambda_i^2) \in [0, Q]^2$ of high-ability students to screen from school S_i. The payoff of school C_i is $U_i(\lambda_A, \lambda_B; q, H_1, H_2)$ defined above. A pair of colleges' screening choices $\lambda_C^* = (\lambda_A^*, \lambda_B^*)$ is a (Nash) equilibrium if $U_A(\lambda_C^*) \geqslant U_A(\lambda_A, \lambda_B^*)$ and $U_B(\lambda_C^*) \geqslant U_B(\lambda_A^*, \lambda_B)$ for all λ_A and λ_B.

As in the first stage, we will assume that the value of Q (the capacity of colleges C_A and C_B) is large enough such that the solutions are interior. More specifically, we have:

Assumption 2. *The colleges' capacities are high enough such that $Q \geqslant \alpha$.*

Under this assumption, the equilibrium of the college admissions subgame is as follows:

Proposition 2. *Under Assumptions 2, there is a unique Nash equilibrium, where the screenings are given by*

$$\left(\lambda_A^{1*}, \lambda_A^{2*}, \lambda_B^{1*}, \lambda_B^{2*}\right) = \left(\frac{H_1}{1+\kappa q}, \frac{H_2}{1+\kappa q}, \frac{H_1}{2+\kappa q}, \frac{H_2}{2+\kappa q}\right)$$

and the equilibrium masses of high-ability students in each school are:

$$\left(H_A^*, H_B^*\right) = \left(\frac{H_1 + H_2}{1+\kappa q}, \frac{\kappa q \left(H_1 + H_2\right)}{\left(1+\kappa q\right)\left(2+\kappa q\right)}\right).$$

Notice that $\frac{H_A^*}{H_B^*} = 1 + \frac{2}{\kappa q}$. That is, the relative advantage that college C_A enjoys over C_B is independent of key parameters in the schools' matching stage, except for the schools' capacities (q). Especially interesting is the fact that the ratio is independent of students' preferences between schools. For example, when σ is close to zero, implying that a large majority of students prefer school S_2 over S_1, college C_A is able to screen those students from S_2 and successfully acquire them at a low cost. This, however, does not translate into an advantage over C_B in terms of equilibrium cohort composition when compared to a situation in which high-ability students are more evenly distributed between the two schools, since the expression is independent of σ.

Notice, moreover, that the equilibrium masses of high-ability students admitted by colleges depend linearly of their masses admitted to each school. More specifically, a proportion $\frac{1}{1+\kappa q}$ of the high-ability students admitted by each school will be admitted to the most-preferred college. This implies that our model, in which schools' preferences are liner in the mass of high-ability students admitted, can also be interpreted as a model in which schools' objective is to maximize the mass of their students who attend their most-preferred college.[4]

Finally, since there is a unique Nash equilibrium in each stage, there is a unique subgame perfect equilibrium as well.

Remark. Under Assumptions 1 and 2, there is a unique subgame perfect Nash equilibrium, characterized by the strategy profiles described in Propositions 1 and 2.

2.5 Strategic students

Up to this point, we have been assuming that students behave *truthfully*: they always apply to both schools and colleges, and when facing offers from multiple institutions, they would simply accept one from their most preferred option. We now relax these assumptions.

[4] Alternatively, the schools' objective could simply be to place as many students into colleges as possible.

First, notice that since students have zero mass, their decisions of not applying to a school or college does not change he schools' screening decisions. As a result, the colleges' decisions remain unchanged as well. Therefore, the only impact that this action would have is to eliminate the possibility of being screened by that institution. This makes the student strictly worse off in expectation. Therefore, students who maximize expected utility will always apply to all schools and colleges. Next, a student could benefit from accepting the offer from a less desirable school in the school admission stage, if that increased the likelihood of her being matched to a more preferred college later on. In contrast, in the college admission stage, it is clear that the (strictly) dominant strategy for every student is to accept the most preferred institution, college C_A.

We say that a student is type $i \in \{1,2\}$ if she prefers school S_i over school S_j where $j \neq i$. Let $u_i(S,C)$ be the payoff for a type i student being matched to school S and college C, where $S \in \{S_1, S_2, \varnothing_S\}$, $C \in \{C_A, C_B, \varnothing_C\}$, and $\varnothing_C$ and $\varnothing_S$ represent not being accepted into any school or college, respectively. Hence, we have $u_i(S, C_A) > u_i(S, C_B)$ for all $S \in \{S_1, S_2, \varnothing_S\}$ and $i \in \{1,2\}$, and $u_i(S_i, C) > u_i(S_j, C)$ for all $C \in \{C_A, C_B, \varnothing_C\}$, $i, j \in \{1,2\}$, and $j \neq i$.

The two-stage admission game, in which schools and colleges interact as before but now students with more than one offer can decide which offer to accept, proceeds as follows:

School admission

- $t = 1$: each school S_i simultaneously chooses the screening value $\lambda_i \in [0, q]$, and offers are sent to the high-ability students screened.
- $t = 2$: *students with offer(s) choose which one to accept, if any.* A school with seats remaining may fill them with students from the internal pool.

College admission

- $t = 3$: each college C_i simultaneously chooses the screening levels λ_i^1 and λ_i^2 for applicants from each school and sends offers to the high-ability students identified.
- $t = 4$: *Students with offer(s) choose one offer to accept, if any.* A college with seats remaining may fill them with students from from the internal pool.

As before, we solve for this game's subgame-perfect Nash equilibrium, and proceed via backward induction. First, in the college admission stage, it is strictly dominant for both types of students to accept at least one offer, when those are given, and to choose college

C_A when facing offers from both colleges, as there is no strategic value to choosing the less-preferred college C_B. Hence, students always accept the offers from college C_A, and only choose college C_B if that is their sole offer. Hence, Proposition 2 holds.

Next, consider the school admission stage. First of all, since colleges only screen students who attended a school, students could never gain anything from not being matched to one. Moreover, a type i student would decline the offer from school S_i to accept the offer from another school only if it increases the expected utility in the college admissions stage, and moreover if that increase in expected utility at least compensates the loss of utility from going to a less-desired school. The proposition below shows, however, that this is never the case in equilibrium

Proposition 3. *In the sequential school and college admission game with strategic students, under Assumptions 1 and 2, the unique subgame perfect Nash equilibrium strategies are described by Propositions 1, 2, and students being truthful.*

Proposition 3 shows that there is no space for strategic behavior for students, with the objective of obtaining better assignments, even when they are almost indifferent between the two schools. The reason for this is that colleges, when optimally deciding the mass of students from each school they will screen, make choices that equate the marginal cost per high-ability student acquired in both schools. The cost of screening for those students, however, is proportional to its marginal scarcity in the pool, and as a result, colleges' screening choices consist of a fixed proportion of high-ability students in each school, making the ratio "mass of students screened from a school"/"mass of high-ability students at a school" constant and equal across the two schools. College C_B, which has a higher marginal cost per high-ability student acquired due to the effect from students' preferences, faces a shift in the cost curve, but the same optimization problem as C_A.

3 Comparative statics and the role of preferences

In this section, we evaluate how screening costs and students' preferences between schools affect schools' and colleges' outcomes and student welfare. Variations in students' preferences are represented by changes in the value of σ. For the purpose of welfare analysis, instead of making cardinal assumptions, we focus on the mass of students who are admitted to their most preferred school and/or college. students.

We will focus on high-ability students' welfare because the channel that affects their welfare is the one that is our main interest: schools and colleges competing for high-ability students through screening. Changes in σ result in changes in the mass of students who are able to exercise their preferences by choosing from multiple offers. Low-ability students, on the other hand, never exercise that choice and therefore their welfare has a "residual" nature: as schools screen less, more seats will be filled with low-ability students. The proportion of those filled by a school who are being matched to their most preferred school is the proportion of the overall population that prefers that school.

3.1 Schools' matching

Proposition 4. *For the schools' admission process:*

(i) the less popular school S_1 admits less high-ability students than the more popular school S_2 ($H_1^* < H_2^*$), and screens less ($\lambda_1^* < \lambda_2^*$). When screening becomes more costly,

(ii) both schools screen less ($\frac{\partial \lambda_1^*}{\partial \kappa} < 0$ and $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$),

(iii) the more popular school S_2 admits less high-ability students ($\frac{\partial H_2^*}{\partial \kappa} < 0$), and

(iv) the less popular school S_1 admits more high-ability students if and only if the screening cost is low: there exists a $\kappa^* > 0$ such that:

$$\begin{cases} \kappa < \kappa^* & \textit{implies } \frac{\partial H_1^*}{\partial \kappa} > 0, \textit{ and} \\ \kappa > \kappa^* & \textit{implies } \frac{\partial H_1^*}{\partial \kappa} < 0. \end{cases}$$

Part (i) of proposition 4 shows that the model is well-behaved and produces natural results. As more students prefer school S_2 over S_1, the expected marginal benefit from screening is higher for the former, because a greater proportion of the students who receive two offers will go there. As a result, S_2 will screen more and obtain more high-ability students in equilibrium. Part (ii) is also natural: higher costs of screening should lead to less screening by both schools.

Perhaps the most intriguing result is the one in part (iv). It shows that when the cost of screening is low enough, an increase in the cost would lead to an increase in the mass of high-ability students acquired by S_1, even though there is also a reduction in the amount of screening done by that school. The reason for this is that the reduction in screening, and the consequent decrease in the expected mass of high-ability students acquired, is more than compensated for by the reduction in the mass of students who also receive an offer from S_2, because a majority of them will reject the offer from S_1. This only happens, however, when the total amount of screening performed by both schools is enough to make the mass of students with two offers high. For this to occur the cost of screening must be low.

A consequence of this result is that since the use of technology allows for a reduction in the cost of screenings, there may be a point at which it will lead to an increase in the competition for each high-ability student, in a way that schools which are lower-ranked will, despite being able to screen more students, see a smaller mass of their offers accepted.

The main results regarding how students' preferences affect outcomes are driven by their effect on schools' equilibrium screening and intake of high-ability students. Remember that we assume, without loss of generality, that $0 \leqslant \sigma < \frac{1}{2}$. Therefore, an increase in σ represents a reduction in the aggregate preference that students have for school S_2.

Proposition 5. *As preferences between schools become more heterogeneous,*

(i) more students receive offers from both schools ($\frac{\partial \lambda_1^* \lambda_2^*}{\partial \sigma} > 0$*),*

(ii) the less popular school admits more high-ability students ($\frac{\partial H_1^*}{\partial \sigma} > 0$*) and the more popular school admits less (*$\frac{\partial H_2^*}{\partial \sigma} < 0$*), and*

(iii) the total mass of high-ability students admitted by any school increases ($\frac{\partial \left(H_1^* + H_2^*\right)}{\partial \sigma} > 0$*).*

Item (ii) of proposition 5 is intuitive: as more students prefer school S_1, the equilibrium mass of high-ability students in S_1 increases and decreases in S_2.

Item (i) is less obvious. Changes in the value of $\lambda_1^* \lambda_2^*$ reflect changes in the mass of students who receive offers from both schools. Students' preferences affect schools' screening choices by changing the expected benefit from screening. As σ increases, the marginal gain from screening increases for school S_1 and decreases for S_2. More specifically, the changes in the marginal gain from screening driven by changes in preferences are as follows:

$$\frac{\partial}{\partial \sigma} \frac{\partial U_1}{\partial \lambda_1} = \frac{\lambda_2}{\alpha} \text{ and } \frac{\partial}{\partial \sigma} \frac{\partial U_2}{\partial \lambda_2} = -\frac{\lambda_1}{\alpha}.$$

By proposition 4, $\lambda_2^* > \lambda_1^*$. Therefore, starting from equilibrium values, the increase in the marginal gain from the screening of school S_1 has a larger magnitude than the decrease in S_2. Therefore, the overall gain from screening increases, leading to a higher overall amount of screening by both schools. As a result, the total mass of admitted high-ability students also increases, as shown in item (iii).

The mass of students who have school S_1 as their most preferred school and receive an offer from S_1 is $\sigma \lambda_1^*$. The mass of those who have school S_2 as their most preferred school and receive an offer from S_2 is $(1 - \sigma) \lambda_2^*$. All the other high-ability students who are admitted are matched to their second choice. So the mass of students who are matched to their most preferred schools is:

$$\mathcal{H}^{1,2} = \sigma \lambda_1^* + (1-\sigma)\,\lambda_2^*.$$

Proposition 6. *As preferences become more heterogeneous, there will be a lower mass and proportion of high-ability students matched to their most preferred school. Let $\mathcal{H}^{1,2}$ be the mass of students who are admitted to their most preferred school among S_1 and S_2. Then,*

$$\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0 \text{ and } \frac{\partial}{\partial \sigma} \frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} < 0.$$

That is, as preferences become less correlated, the mass (and proportion) of high-ability students who are matched to their top choice decreases. At first sight this may seem unintuitive. After all, when preferences are less correlated there is less competition between students for the schools' seats. Moreover, as shown in Proposition 5, there is an overall increase in the total amount of screening, so that more students receive offers from both schools. To better understand the source of that result, it is useful to disentangle the two effects that the shift in students' preferences has on $\mathcal{H}^{1,2}$:

$$\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} = \underbrace{\sigma \frac{\partial \lambda_1^*}{\partial \sigma} + (1-\sigma)\frac{\partial \lambda_2^*}{\partial \sigma}}_{\text{Screening change effect}} - \underbrace{\lambda_2^* - \lambda_1^*}_{\text{Screening gap effect}}.$$

The *screening change effect* relates to the fact that a change in σ leads to an increase in λ_1^* and a decrease in λ_2^*. The former produces an increase in the mass of students who prefer and receive an offer from S_1, while the latter produces a reduction in the mass of students who prefer and receive an offer from S_2. The *screening gap effect*, on the other hand, relates to the fact that the increase in σ implies that more students prefer a school that screens fewer students than one that screens more. The result shows that even when the screening change effect increases the mass of students matched to their preferred school, this is dominated by the screening gap effect.

3.2 Colleges' matching

The results from Proposition 5 produce some important effects on colleges' outcomes as well, outlined in the proposition below:

Proposition 7. *As preferences between schools become more heterogeneous, colleges admit more high-ability students. In other words, $\frac{\partial H_A^*}{\partial \sigma} > 0$ and $\frac{\partial H_B^*}{\partial \sigma} > 0$.*

Proposition 7 shows that, as opposed to schools, both colleges obtain a better set of students when preferences are more heterogeneous. This happens because colleges are able to "free ride" on schools' increased screening in reaction to the change in preferences.

As students have common preferences between the colleges, the mass of students who are matched to their top choice is simply the mass of students matched to college C_A. We now proceed to evaluate overall welfare by combining the two levels of education.

The proportion of high-ability students in schools S_1 and S_2 who are at their top choice among schools are $\frac{\sigma\lambda_1^*}{H_1^*}$ and $\frac{(1-\sigma)\lambda_2^*}{H_2^*}$, respectively. Since the mass of students that college C_A admits from each of the schools equals the mass of those who are screened, the total mass of students who are matched to their top school and are then matched to C_A (their top choice among colleges) is:

$$\mathcal{H}^{A,B} = \lambda_A^{1*}\sigma\frac{\lambda_1^*}{H_1^*} + \lambda_A^{2*}\left(1-\sigma\right)\frac{\lambda_2^*}{H_2^*}.$$

We know that $\lambda_A^{1*} = \frac{H_1^*}{1+\kappa q}$ and $\lambda_A^{2*} = \frac{H_2^*}{1+\kappa q}$. Let $\mathcal{H}_1^{A,B}$ be the value of $\mathcal{H}^{A,B}$ in that case:

$$\mathcal{H}_1^{A,B} = \frac{\sigma\lambda_1^* + \left(1-\sigma\right)\lambda_2^*}{1+\kappa q} = \frac{\mathcal{H}^{1,2}}{1+\kappa q}.$$

The effect that changes in preferences have on $\mathcal{H}^{A,B}$ is, therefore:

Proposition 8. *As preferences between schools become more heterogeneous, there will be a smaller mass and proportion of high-ability students matched to their most preferred school and college. Let $\mathcal{H}^{A,B}$ be the mass of students who are admitted to their most preferred school and college. Then:*

$$\frac{\partial\mathcal{H}^{A,B}}{\partial\sigma} < 0 \text{ and } \frac{\partial}{\partial\sigma}\frac{\mathcal{H}^{A,B}}{H_1^{A*}+H_2^{A*}+H_1^{B*}+H_2^{B*}} < 0.$$

The result above shows that although that change in preferences increases the absolute mass of students matched to both colleges, and as a consequence of those matched to their top college C_A (Proposition 7), that increase is more than compensated for by a decrease in the mass of students who are matched to their most preferred school.

4 Sources of inefficiency and policy recommendations

Our model, and the results that we obtain from it, point at multiple sources of inefficiency that result from decentralized and competitive screenings performed by schools and colleges.

In this section we discuss each one of them, interpret their relationship with our results, and evaluate alternative policies to mitigate them. When comparing the welfare of different alternatives, we will consider the utilitarian aggregation of students' utilities separately from that of the institutions. Doing this prevents arbitrary considerations when comparing screening costs and students' cardinal utilities.

4.1 Sources of inefficiency

Students' preference mismatch The first source of inefficiency lies in the fact that, in equilibrium, both schools will admit students that are matched to their less preferred schools. This happens when a student is screened only by her less preferred school, and therefore does not have the option of choosing an offer from her most preferred one instead. Pairwise exchanges of students who would prefer each other's school, as long as they involve students who are both high ability or from the internal pool, would constitute Pareto improvements.[5] Notice, however, that none of these exchanges are possible between the two colleges: moving any student from college C_A to C_B would make them worse off.

Since not every student receives offers from both schools in equilibrium and these offers are independent of their preferences, there is always, in equilibrium, a positive mass of students who could be made better off through these exchanges. Moreover, as shown in Proposition 6, the proportion of high-ability students who are admitted to their most preferred school is reduced when preferences are more heterogeneous. Notice that these are the scenarios in which Pareto improving exchanges are more present.

Screening overlap In our model, whether a student is high ability or not is not known by any individual or institution unless that fact is determined via the screening process. Once that is done by some school, that information becomes known by the school that made the identification. Therefore, any overlap in the schools' (and colleges') screening is wasteful, in the sense that this identification is costly, but no new information is obtained by screening the same student twice.

The overlap also distorts schools' incentives to screen, in that it shifts the returns to screening to be lower than in its absence, but only to the extent that students with another offer would choose the other school.

[5] Pareto improvements are changes in an allocation in which some agents are made strictly better off, and no agent is made worse off. An allocation is Pareto efficient if no other (feasible) allocation Pareto improves upon it.

We show in Proposition 4 that, as expected, as the cost of the screening procedures is reduced, the overall screening level increases, but at the same time so does the wasteful screening associated with the overlap. This is also the case when preferences become more heterogeneous (Proposition 6).

Colleges' screening We have shown that, because schools obtain in equilibrium different proportions of high-ability students in their cohorts, colleges use the information contained in the different composition of these cohorts to improve the efficiency of their screening costs. This improvement, however, does not change the fact that the screening performed by colleges is wasteful, in that schools have already identified every high-ability student in their cohorts. That is, while information transmission from schools to colleges driven by students' preferences makes colleges' screening more efficient, there is still potential for improvements if information can be exchanged between schools and colleges.

4.2 Policy interventions

4.2.1 Preventing an overlap

The overlaps in the screening of the external pool come from the fact that both schools independently screen from the same pool of applicants. We can think of two methods for preventing this from happening. One is to have only one institution perform the screening. The other is to partition the pool of applicants so that each school screens only from a subset of the students. In both cases, we will assume that the screening technology is the same as the one used by the schools in our main model.

Consider first the option of centralized screening. The objective of a centralized procedure that maximizes the sum of the schools' utilities is the following:

$$\max_{\lambda}\left[2\lambda-\kappa\int_{0}^{2\lambda}\frac{x}{\alpha-x}dx\right]$$

The optimal screening level is $\hat{\lambda}=\frac{\alpha}{2+2\kappa}$, and therefore the mass of high-ability students who would be introduced into the schooling system is $\hat{H}=\frac{\alpha}{1+\kappa}$.

Next, consider the alternative of partitioning the applicants into two pools, which can be screened independently by both schools. Let θ be the share of the external pool that is allocated to a school. The school's optimization problem is:

$$\max_{\lambda}\left[\lambda-\kappa\int_0^{\lambda}\frac{\theta x}{\theta\alpha-x}dx\right]$$

The optimal screening for that school is, therefore, $\hat{\lambda}=\frac{\theta\alpha}{1+\theta\kappa}$. When partitioning the students so that one school screens a mass θ and the other the remaining $1-\theta$, the mass of high-ability students in both schools is:

$$\hat{H}=\frac{\theta\alpha}{1+\theta\kappa}+\frac{(1-\theta)\alpha}{1+(1-\theta)\kappa}$$

One can easily confirm that $\hat{H}$ is maximized when $\theta=\frac{1}{2}$. Partitioning the mass of students equally between the schools, therefore, maximizes the screening. In this case, $\hat{H}=\frac{2\alpha}{2+\kappa}$. That is, the total mass of high-ability students admitted into schools is strictly larger than when the screening process takes place centrally. This fact is not especially surprising, since screening costs are convex, and therefore one institution screening for all schools will reach much higher marginal costs.

What is surprising, perhaps, is that one can check that even this value of $\hat{H}$ is lower than the equilibrium mass of high-ability students that is obtained in our main model ($H_1^*+H_2^*$ in proposition 1). To understand the driver behind this, notice that the marginal cost of screening from a mass of applicants with a share θ of the external pool is the following:

$$\frac{\partial\mathcal{C}}{\partial\lambda}=\frac{\lambda\theta}{\alpha\theta-\lambda}$$

Since $\frac{\lambda\theta}{\alpha\theta-\lambda}>\frac{\lambda}{\alpha-\lambda}$ for any $0<\theta<1$ and $\lambda>0$, the marginal cost of screening smaller pools is higher, even if the proportion of high-ability students is the same. This is because after screening a certain mass λ from a pool, the one with a smaller mass becomes more *diluted*: the proportion of high-ability students who remain in the pool is smaller than compared with a larger pool. Therefore, the likelihood that the next student drawn from the smaller pool is high ability is smaller.

We have, therefore, a trade-off: partitioning the applicants between the schools eliminates the losses that come from the overlap, but it makes the screening process more costly, since the pools from which schools will screen become more diluted, when compared with the alternative formulation.

4.2.2 Preventing colleges from screening again

In our model, colleges have no direct access to the results of the screening performed by the schools. Since this information is available to schools and valuable to colleges, there exists a clear opportunity for a market for that information to emerge. The question we try to answer in this section is whether a screening process that is coordinated by some entity (such as an education department) which prevents a screening overlap by partitioning the pools of applicants, and allows schools to sell information about their students to colleges, could lead to a larger proportion of high-ability students entering the education system. In this subsection, therefore, we might not make considerations about the total expenditure in screening.

Consider our model in the previous section, in which schools screen from optimally partitioned pools of applicants (that is, $\theta = \frac{1}{2}$), and in which colleges pay ω per identification of a high-ability student. We will assume that schools will not sell information about the same student to both colleges, and that colleges will not perform independent screening anymore. Notice that both colleges will pay for information about every student if $\omega \leqslant 1$, but will not pay if $\omega > 1$. The optimization problem for each school therefore becomes:

$$\max_{\lambda} \left[\lambda(1+\omega) - \kappa \int_0^{\lambda} \frac{x}{\alpha - 2x} dx \right]$$

The optimal screening level is $\hat{\lambda} = \frac{(1+\omega)\alpha}{2+2\kappa}$, and the mass of high-ability students who would be introduced into the schooling system is $\hat{H} = \frac{2(1+\omega)\alpha}{2(1+\omega)+\kappa}$.

Clearly, for any $\omega > 0$, the total mass of students admitted to the education system is larger than the case in which schools simply screened from their shares of the pool without any secondary market for information. The next question is whether the outcome from this intervention can place more high-ability students in schools and colleges than under the decentralized screening process.

When considering a utilitarian aggregation of schools and colleges, the optimal value of ω is 1, and therefore $\hat{H} = \frac{4\alpha}{4+\kappa}$. If we compare that with the equilibrium mass of high-ability students that is obtained in our main model in Proposition 1, we find that the following condition is necessary and sufficient for this alternative policy to result in a higher mass of high-ability students in schools and colleges:

$$\kappa > \sqrt{2\left(2 + \sigma(1-\sigma)\right)} - 2$$

The expression on the right-hand side of the inequality above is maximized when $\sigma \to \frac{1}{2}$.

Therefore, $\kappa > \frac{1}{2}\left(3\sqrt{2}-4\right) \approx 0.121$ is a sufficient condition for the screening process without overlap and with a market for information to improve upon the decentralized process, in terms of the mass of high-ability students in the market.

4.2.3 Improving students' welfare

Our next policy intervention relates to the students' welfare. The question we pose is the following: fixing the masses of high-ability students in each school and college, what is an efficient allocation of students among these institutions, and how could that be implemented?

The first thing to note is that any allocation of students between colleges that fill their capacities is Pareto efficient, since every student in college C_A is allocated to her most preferred college, and therefore any exchange between colleges would make one student worse off.

Consider next the allocation of students among schools, and let $\hat{H}_1$ and $\hat{H}_2$ be the masses of high-ability students allocated to schools S_1 and S_2, respectively. If we are able to select the students in each one of these sets based on their preferences, the Pareto efficient allocations simply maximize the mass of students assigned to their most preferred school. So if $\hat{H}_1 \leqslant \sigma$ and $\hat{H}_2 \leqslant 1-\sigma$, all of the students could be assigned to their most preferred school.[6] If $\hat{H}_1 > \sigma$ or $\hat{H}_2 > 1-\sigma$ (it is impossible for both to be true), then one school will have only students who prefer it, while the other will have some high-ability students who do not.

Notice, however, that in order for a policy maker to implement these assignments, students would have to reveal their preferences. If the likelihood of being screened (and therefore being assigned to a school), conditional on each preference revealed, is distinct for the two possible preference rankings, then whether students would truthfully reveal that information would depend on their cardinal preferences.[7] While an analysis of this more general implementation problem is itself interesting, we consider instead the case in which the policy maker guarantees that a student's likelihood of being screened is independent of their preference. In this case, given a screening level $\hat{\lambda}$ that could be implemented by partitioning the external pool between schools or centrally, masses $\hat{\lambda}\sigma$ and $\hat{\lambda}(1-\sigma)$ of students prefer schools S_1 and S_2, respectively. A centralized clearinghouse could then be used to determine which school each student will attend. By maximizing the mass of those who are matched to their most preferred school, and

[6]This also includes the students from the internal pool. Since these are arbitrarily large, there are always enough students who prefer a school to fill the seats taken by unscreened students.

[7]Intuitively speaking, if the likelihood of being screened is higher when a student reveals a false preference, and the assignment is Pareto efficient for the preferences revealed, then deviating from the true preference might be profitable when the difference in cardinal utility between the two schools is small, but not if it's large.

making their assignment between colleges independent of their preferences between schools, we could guarantee that students would have an incentive to truthfully reveal their preference, and this "second-best" allocation could be implemented.

Notice that for any given positive mass of students screened, this procedure maximizes the mass of students matched to their most preferred school. Since in a decentralized procedure the screening of a student is also independent of their preference, the use of this preference elicitation procedure together with the "second-best" allocation is a strict Pareto improvement over the decentralized outcome for any given mass of allocated high-ability students.

4.2.4 Application costs and signalling

Throughout this paper, we assumed that students incur no cost in applying to schools and colleges. As we argued in Section 1, this allowed us to isolate the effect of the schools' screening overlap as a channel through which students' preferences affect outcomes and schools' incentives. Moreover, since students do not know their abilities when applying for schools, we could not induce different behavior from students with different abilities.

A student who receives an offer from a school, however, might be able to infer her ability from that fact, and therefore application costs for colleges or some other signaling method could serve as a way for high-ability students to credibly indicate their abilities. Given application costs, colleges could choose a screening level that would eliminate the incentive for low-ability students to apply.

To see this, let us assume that students know their ability levels even when applying to schools. Moreover, let u^{top} and u^{sec} be respectively the cardinal utilities from attending their most-preferred school and least-preferred school, γ be an exogenously set application cost, charged at each school,[8] and as before λ_i represent the screening level chosen by school S_i. There exists a value κ^*, such that for every $\kappa < \kappa^*$, setting $\gamma = u^{sec}$ and λ_1^* and λ_2^* as below constitutes a separating perfect Bayesian Nash Equilibrium of the subgame played at the school admissions level.

$$\lambda_1 = \frac{u^{top} - u^{sec}}{u^{top}}\sigma \quad \text{and} \quad \lambda_2 = \frac{u^{top} - u^{sec}}{u^{top}}(1 - \sigma)$$

In this equilibrium, both schools screen only a mass λ_i of their applicants, rejecting eventual low-ability students identified in this process, and admitting the remaining applicants from the external pool without screening. Notice that, since $u^{top} > u^{sec}$, schools screen less

[8] We assume that application costs are used only for screening and do not enter the schools' utility functions.

students than they admit. High-ability students only apply for their most-preferred school, and low-ability students will not apply to any. Next, we show that no student or school can profitably deviate from this profile.

Since $\gamma = u^{sec}$, high-ability students apply to their most-preferred schools and are accepted, obtaining utility $u^{top} - u^{sec}$. Applying to both schools or to the least-preferred school would clearly reduce their utilities, and therefore high-ability students have no profitable deviation. Consider next the low-ability students. The probability that a student who applies to school S_1 is screened is $\frac{\lambda_1}{\sigma}$. Therefore, the expected utility of one who deviates and applies to that school is at most $u^{top}\left(1 - \frac{\lambda_1}{\sigma}\right) - \gamma$. Using the expressions for λ_1 and γ above, we find that the expected utility that one of these students obtains is zero, and therefore they would not deviate, sustaining the equilibrium. Finally, school S_1 would not deviate by reducing the value of λ_1, because that would lead every low-ability student to apply, without increasing the number of high-ability ones. And as long as κ is not too high—which is given by our assumption about its value—the screening levels λ_i would still lead to positive utility for the school.

A similar reasoning could be used to extend this use of application costs at the college level. We see, therefore, that under certain assumptions about the screening costs and students' information about their own abilities, an assignment of students to schools and colleges satisfying our welfare objectives could be attained with lower screening levels, and some loss of utility from the part of students in the form of application costs.

Finally, an alternative solution for the inefficiencies that derive from competitive screening is the development of a signalling device [Spence, 1973], such as an exam that students can choose to take. If high-ability students incur in a lower effort to obtain high exam grades than low-ability ones, under certain conditions on the students' utility functions, there will be here also a separating equilibrium, in which only high-ability students take the exam and obtain a grade that is high enough for the schools to admit them by simply observing their exam grade without screening them. If the cost of this effort on the part of students is low enough, this solution could be superior to the competitive screening that we consider in our main model.

5 Robustness

In this section, we discuss the impact in our results of the assumption of linear ordering of screening methods. When describing the cost that schools and colleges incur when identifying high-ability students, we assume that the different methods have costs that can be ordered linearly. Notice that this does not imply that screening costs themselves are linear.

Suppose that instead of a linear ordering κx, we use a generic function $h(x)$, where $h(0) = 0$. The marginal cost of screening a mass λ of high-ability students would then be:

$$\frac{\partial \mathcal{C}}{\partial \lambda} = \frac{h(\lambda)\eta}{p_h - \lambda}$$

One can easily check that $h' \geqslant 0$ is a sufficient condition for the screening cost to be convex in λ over the entire domain. Notice, however, that $h' \geqslant 0$ by construction, since it represents the costs of the screening methods when ordered by increasing cost. Therefore, the cost function will always be such that marginal costs are increasing, and solutions will be interior.

As long as screening costs are convex and, as we do in our main analysis, we restrict our attention to interior solutions, the difference in popularity among schools and colleges will guarantee a gap between the marginal utilities from screening. Since less popular schools will see more of their offers rejected, their marginal gain from screening is lower than the more popular one, and therefore the results that rely on this fact — such as items (i), (ii), and (iii) of Proposition 4, Proposition 5, and therefore Proposition 7 — will still hold. On the other hand, the results in item (iv) in Proposition 4, Proposition 6, and Proposition 8 depend more specifically on the function h, and therefore the determination of whether they hold or not would depend on a closer analysis of each case.

Proposition 3 is perhaps where the linearity of h has the strongest qualitative consequences. To see this, consider the first-order conditions for the screening choices of college A when the ordering of screening costs is given by a function h:

$$h'\left(\lambda_A^{1*}\right) q = H_1^* - \lambda_A^{1*} \quad (5.1)$$

$$h'\left(\lambda_A^{2*}\right) q = H_2^* - \lambda_A^{2*} \quad (5.2)$$

When $h'(x) = \kappa x$, as in our baseline model, the two equations above imply that $\frac{\lambda_A^{1*}}{H_1^*} = \frac{\lambda_A^{2*}}{H_2^*} = \frac{1}{1+\kappa q}$. That is, the probability that a high-ability student receives an offer from college C_A is the same, regardless of which school she is coming from. This fact explains why students do not have incentive to accept an offer from a less desired school, and Proposition 3 has a unique truthful equilibrium: she will lose utility from her school and not improve her likelihood of being accepted at the most preferred college.

More generally, as long as $\frac{\lambda_A^{1*}}{H_1^*} = \frac{\lambda_A^{2*}}{H_2^*}$ and $\frac{\lambda_B^{1*}}{H_1^*} = \frac{\lambda_B^{2*}}{H_2^*}$, students do not benefit from being strategic when choosing schools. The functions h for which this is true depends, however, not only on colleges' first-order conditions, but also on how changes in H_1^* and H_2^* are related. These in turn also depend on the h function, since h is involved in schools' choices as well, and potentially also on students' incentives to strategically choose which schools to accept.

Consider as an example the case when $h(x) = x^2$. By solving college C_A's first-order condition, we see that the values of λ_A^{1*} and λ_A^{2*} are such that $\frac{\lambda_A^{1*}}{H_1^*} > \frac{\lambda_A^{2*}}{H_2^*}$ if and only if $H_1^* < H_2^*$. That is, high-ability students coming from schools with a lower proportion of similarly able students have a higher probability of receiving an offer from college C_A.[9] In isolation, this fact would imply that depending on how students' utility functions are, they could have incentives to strategically accept an offer from a least preferred school, in order to obtain better chances of receiving an offer from college C_A.

This observation is only partial, however. Given these incentives, the values of H_1^* and H_2^* themselves would be endogenous to students choices that do not depend only on σ, since the assumption that students simply follow their preferences would not hold anymore. This, in turn, would change the marginal utility from schools' screening, in the sense that the expected mass of students who would accept an offer depends on solving new first-order conditions that now involve the strategies of the students who have incentives to simply not follow their preferences. Even in the absence of these issues, however, the quadratic h already makes the equilibrium expressions intractable for the qualitative analysis that we make in section 3.

All of this indicates, therefore, that alternative formulations for the function h might have an impact on the tractability of the model and on students' incentives that render the analytical evaluation of the strategic interaction between the schools' and colleges' screening essentially intractable, to the best of our knowledge.

Notice, however, that the assumption that we use, and that allows us to obtain valuable insights from the closed-form solutions, is not based on particular arbitrary assumptions. One way to interpret $h(x) = \kappa x$ is that the cost of any screening technology developed in the market is ex-ante uniform between a lower-bound of 0 and some upper-bound value, which is never reached by the schools and colleges in our model. Alternative shapes for $h()$ would imply that some costs are more likely than others.

[9] The reason behind this seemingly unintuitive result is that since the cost of the screening methods increases convexly, the reduction in the amount of optimal screening will be larger for the school that has the largest absolute screening under the linear h function. Therefore, while the mass of offers to students in school S_1 is still smaller, the proportion of high-ability students with offers is larger.

Finally, one last possibility that one can consider for the function $h(x)$ is simply the case where $h(x) = 0$. That is, what would happen if schools and colleges could screen for high-ability students costlessly? It is not hard to see that this model would lose its ability to produce valuable insights into any of the issues that we evaluated. Given Assumption 1, the marginal gain from screening is always positive, and both schools would screen all students from the external pool. As a result, the screening overlap will contain all high-ability students, every student will receive offers from both schools, and accept the offer from their most preferred school. Therefore, every high-ability student will be admitted into a school, and all of them to their most-preferred school. Also, under Assumption 2, they will all go to college C_A, and college C_B will be indifferent between screening or not, since it cannot admit any high-ability student. Differently from most of our results, none of these facts would change when students' preferences change.

6 Conclusion

In this article, we evaluate the effect that competitive screening for high-ability individuals has on the matching of students to schools and colleges, focusing on three aspects of this process. First, we show how the lack of coordination between schools' screening decisions results in an overlap between the individuals identified as high ability by those institutions. This overlap creates a channel through which students' preferences interact with the expected return from screening: the less popular school will see most of their offers that overlap with the more popular school rejected, which in turn increases the cost of obtaining its students. Under convexity of screening costs, therefore, the aggregate incentive for screening students is the highest when students' preferences are more homogeneous. Interestingly, however, this does not translate into higher student welfare: as preferences become less correlated, the reduction in the mass of students who are screened by their most preferred school is greater than the increase in choices that come from a larger overlap.

Next, we consider the fact that the screening performed by schools generates information that can be used by colleges that select students from those schools. Because more popular schools have a lower cost of obtaining these students, students coming from them have a higher probability of being high-ability. Moreover, this creates a positive externality from a more competitive landscape between schools: the more homogeneous preferences between schools are, the more schools need to screen for high ability students, increasing the quality of the pool of college applicants.

Other than providing a better understanding of the incentive and welfare effects that competitive screening and the information it produces, our results may provide some guidance for improving these processes. One is that improvements in schools' screening technologies may, at a certain point, be detrimental to low-ranked schools (Proposition 4), increasing the gap between them. Another one is that investments on reducing the gap between higher and lower-ranked schools may increase the competition between them and not only increase the overall mass of high-ability students absorbed into the education system, but also generate positive externalities further down the education stream (Propositions 5 and 7).

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Appendix

Proofs of Propositions

Proof of proposition 1

Schools 1 and 2's best-response functions are as follows:

$$\lambda_1 = \frac{\alpha\left(\alpha - \lambda_2(1-\sigma)\right)}{\alpha(1+\kappa) - (1-\sigma)\lambda_2}, \text{ and } \lambda_2 = \frac{\alpha\left(\alpha - \lambda_1\sigma\right)}{\alpha(1+\kappa) - \sigma\lambda_1}$$

Notice that $\frac{\partial\lambda_1}{\partial\lambda_2} < 0$ and $\frac{\partial\lambda_2}{\partial\lambda_1} < 0$, that is, schools' screenings are strategic substitutes. Therefore, the maximum value for λ_1 is produced when $\lambda_2 = 0$, in which case $\lambda_1 = \frac{\alpha}{1+\kappa} < \alpha$. Screening decisions are thus always interior. The same holds for λ_2 . By solving the system, we obtain the equilibrium screening λ^* and equilibrium masses of high-ability students H_1^* and H_2^*. Under Assumption 1, we have that $H_i^* < q$.

Proof of proposition 2

Colleges' best-response functions are as follows:

$$\left(\lambda_A^1, \lambda_A^2, \lambda_B^1, \lambda_B^2\right) = \left(\frac{H_1}{1+\kappa q}, \frac{H_2}{1+\kappa q}, \frac{H_1\left(H_1 - \lambda_A^1\right)}{H_1^*\left(1+\kappa q\right) - \lambda_A^1}, \frac{H_2\left(H_2 - \lambda_A^2\right)}{H_2\left(1+\kappa q\right) - \lambda_A^2}\right).$$

There is a unique solution to this system of equations. By solving it, we obtain the equilibrium values λ_C^*, and the equilibrium masses of high-ability students in each college H_A^* and H_B^*. Under Assumption 2, we have that $H_A^*, H_B^* < Q$.

Proof of proposition 3

Consider first the college admissions stage. As mentioned in the text, students facing offers from colleges have the dominant strategy of accepting the most preferred offer received, if any. Therefore, given the values of the masses of high-ability students in each college H_1 and H_2, the equilibrium screening values for both colleges is the one given by Proposition 2. This solves the Nash equilibrium of the college admissions subgame, for any values of H_1 and H_2.

Consider next a student of type $\theta \in \{1, 2\}$ facing two offers in the school admissions stage,[10] and let $P\left(C_i|S_j\right)$ denote the probability that the student will be matched to college C_i after attending school S_j.

[10] As we mentioned before, rejecting all offers is never a best-response for the students at any stage.

The expected utility of that student accepting an offer from school S_j is therefore given by:

$$\begin{aligned}P\left(C_A|S_j\right)u_\theta(S_j,C_A)+\left(1-P(C_A|S_j)\right)P\left(C_B|S_j\right)u_\theta(S_j,C_B)\\ +\left(1-P(C_A|S_j)\right)\left(1-P(C_B|S_j)\right)u_\theta(S_j,\varnothing_C)\end{aligned}$$

Therefore, a type θ student will accept the offer from school S_i over one from S_j when:

$$\begin{aligned}&P\left(C_A|S_i\right)u_\theta(S_i,C_A)+\left(1-P(C_A|S_i)\right)P(C_B|S_i)u_\theta(S_i,C_B)+\\ &+\left(1-P(C_A|S_i)\right)\left(1-P(C_B|S_i)\right)u_\theta(S_i,\varnothing_C)\\ \geqslant &P\left(C_A|S_j\right)u_\theta(S_i,C_A)+\left(1-P(C_A|S_j)\right)P(C_B|S_j)u_\theta(S_j,C_B)\\ &+\left(1-P(C_A|S_j)\right)\left(1-P(C_B|S_j)\right)u_\theta(S_j,\varnothing_C)\end{aligned}$$

Since high-ability students from each school are equally likely to be screened, the probability that one of these students, who attended school S_j, is screened by college C_i, is $P\left(C_i|S_j\right)=\frac{\lambda_i^j}{H_j}$. By Proposition 2, this implies that:

$$\begin{aligned}P\left(C_A|S_1\right)=P\left(C_A|S_2\right)&=\frac{1}{1+\kappa q}\\ P\left(C_B|S_1\right)=P\left(C_B|S_2\right)&=\frac{1}{2+\kappa q}\end{aligned}$$

Therefore, a type θ student will accept the offer from school S_i over one from S_j when:

$$\begin{aligned}&u_\theta(S_i,\varnothing_C)-u_\theta(S_j,\varnothing_C)+P(C_A)[u_\theta(S_i,C_A)-u_\theta(S_j,C_A)]\\ \geqslant &\left(1-P(C_A)\right)P(C_B)[u_\theta(S_j,C_B)-u_\theta(S_i,C_B)].\end{aligned}$$

It is easy to see that when $\theta=i$, the left side of the expression is positive but the right side is negative. This implies that truth-telling is a always a best-response for both types of students.

Since students being truthful is always a best-response, Proposition 1 also holds in equilibrium, and therefore the strategies in this unique subgame perfect Nash equilibrium are those described by Propositions 1 and 2 for schools and colleges, and truthful behavior for the students.

Proposition 4

Item (i)

First, we want to show that $H_1^* < H_2^*$. Suppose, on the contrary, that

$$\frac{H_1^*}{H_2^*} = \left(\frac{1-\sigma}{\sigma}\right) \frac{(\kappa(\kappa+2) - A + \sigma(\sigma+1))}{(\kappa(\kappa+2) - A + (\sigma-3)\sigma + 2)} \geqslant 1,$$

which implies $(1-2\sigma)\left(\kappa(\kappa+2) + (\sigma-1)\sigma - A\right) \geqslant 0$. Given $\sigma < \frac{1}{2}$, we have $\kappa(\kappa+2) + (\sigma-1)\sigma \geqslant A$. Using the definition of A, we have $8\kappa(1-\sigma)\sigma \leqslant 0$, which is a contradiction. Therefore, $H_1^* < H_2^*$.

Second, we want to show that $\lambda_1^* < \lambda_2^*$. Suppose not. Then

$$\frac{\lambda_1^*}{\lambda_2^*} = \frac{(\kappa-\sigma+1)(1-\sigma)\left(\kappa(\kappa+2) - A + \sigma(\sigma+1)\right)}{(\kappa+\sigma)\sigma\left(\kappa(\kappa+2) - A + (\sigma-3)\sigma + 2\right)} \geqslant 1,$$

which implies $(1-2\sigma)\left(\kappa^3 + 3\kappa^2 - \kappa\left(A - \sigma^2 + \sigma - 2\right) - A - \sigma^2 + \sigma\right) \geqslant 0$. Given that $\sigma < \frac{1}{2}$, we have $(1+\kappa)A \leqslant \kappa^3 + 3\kappa^2 + \kappa\left(2 - \sigma + \sigma^2\right) - \sigma(1-\sigma)$. Using the definition of A, we have $4\kappa(1-\sigma)\sigma\left(\kappa(1+\kappa) + \sigma(1-\sigma)\right) \leqslant 0$, which is a contradiction. Therefore, we have $\lambda_1^* < \lambda_2^*$.

Item (ii) First, note that $\frac{\partial A}{\partial \kappa} = \frac{4\kappa^3 + 12\kappa^2 + 4\kappa(2-\sigma(1-\sigma)) + 4\sigma(1-\sigma)}{2A}$. We separate the proof into two parts: (a) : $\frac{\partial \lambda_1^*}{\partial \kappa} < 0$; and (b) : $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$.

Part (a): Recall that $\lambda_1^* = \frac{\alpha}{2\sigma} \frac{\kappa(\kappa+2) + \sigma(\sigma+1) - A}{(\kappa+\sigma)}$. We then have:

$$\frac{\partial \lambda_1^*}{\partial \kappa} = \frac{\alpha}{2\sigma} \frac{(\kappa+\sigma)\left(2\kappa + 2 - \frac{\partial A}{\partial \kappa}\right) - \left(\kappa\left(\kappa+2\right) + \sigma\left(\sigma+1\right) - A\right)}{\left(\kappa+\sigma\right)^2}.$$

Hence, it suffices to show that (i): $2\kappa + 2 - \frac{\partial A}{\partial \kappa} < 0$ and (ii):$\kappa\left(\kappa+2\right) + \sigma\left(\sigma+1\right) - A > 0$.

For (i), note that $2\kappa + 2 - \frac{\partial A}{\partial \kappa} = 2\kappa + 2 - \frac{4\kappa^3 + 12\kappa^2 + 4\kappa(2-\sigma(1-\sigma)) + 4\sigma(1-\sigma)}{2A}$ so that it suffices to show that $4\left(\kappa+1\right)A \geqslant 4\kappa^3 + 12\kappa^2 + 4\kappa\left(2 - \sigma\left(1-\sigma\right)\right) + 4\sigma\left(1-\sigma\right)$. This is equivalent to $64\kappa\sigma\left(1-\sigma\right)\left(\kappa^2 + \kappa - \sigma^2 + \sigma\right) \geqslant 0$, which always holds because $-\sigma^2 + \sigma \geqslant 0$, as $\sigma \in \left(0, \frac{1}{2}\right)$.

For (ii), note that $\kappa\left(\kappa+2\right) + \sigma\left(\sigma+1\right) > A \geqslant 0$ is equivalent to $\left(\kappa\left(\kappa+2\right) + \sigma\left(\sigma+1\right)\right)^2 > \kappa^4 + 4\kappa^3 + 2\kappa^2\left(2 - \sigma\left(1-\sigma\right)\right) + 4\kappa\sigma\left(1-\sigma\right) + \left(1-\sigma\right)^2\sigma^2$, which can be simplified as $4\sigma\left(\kappa+\sigma\right)^2 > 0$, which is always true.

Part (b) . Recall that $\lambda_2^* = \frac{\alpha}{2\sigma} \frac{\kappa(\kappa+2) + \sigma(\sigma-3) + 2 - A}{(\kappa+1-\sigma)}$, we have

$$\frac{\partial \lambda_2^*}{\partial \kappa} = \frac{\alpha}{2\left(1-\sigma\right)} \frac{\left(\kappa+1-\sigma\right)\left(2\kappa + 2 - \frac{\partial A}{\partial \kappa}\right) - \left(\kappa\left(\kappa+2\right) + \sigma\left(\sigma-3\right) + 2 - A\right)}{\left(\kappa+1-\sigma\right)^2}.$$

Note that we have proved in part (a) that $2\kappa + 2 - \frac{\partial A}{\partial \kappa} < 0$. Hence, $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$ holds if $\kappa(\kappa+2) + \sigma(\sigma-3) + 2 > A$. This is equivalent to $(\kappa(\kappa+2) + \sigma(\sigma-3) + 2)^2 > \kappa^4 + 4\kappa^3 + 2\kappa^2(2 - \sigma(1-\sigma)) + 4\kappa\sigma(1-\sigma) + (1-\sigma)^2\sigma^2$ which can be simplified as $4(1-\sigma)(\kappa - \sigma + 1)^2 > 0$, which is always true.

Item (iii) First note that $H_2^* = \frac{\alpha\left(\kappa^2 - A - \sigma(1-\sigma)\right)(\kappa(\kappa+2) - A + \sigma(\sigma-3)+2)}{4(1-\sigma)(\sigma-1-\kappa)(\kappa+\sigma)} = \frac{\left(A + \sigma(1-\sigma) - \kappa^2\right)}{2(\kappa+\sigma)}\lambda_2^*$. We can show, through some algebra, that $\frac{\partial H_2^*}{\partial \kappa} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -2\kappa^5 - 6\kappa^4 - \kappa^3\left(6 + 4\sigma - 4\sigma^2\right) - \kappa^2\left(3 + 5\sigma - 6\sigma^2\right) - 2\kappa\sigma\left(4 - 9\sigma + 8\sigma^2 - 3\sigma^3\right) - (1-\sigma)^2\sigma, \text{ and}$$
$$Y = 2\kappa^7 + 10\kappa^6 + 2\kappa^5\left(9 + \sigma - \sigma^2\right) + \kappa^4\left(13 + 17\sigma - 16\sigma^2\right) + \kappa^3\left(2 + 32\sigma - 44\sigma^2 + 24\sigma^3 - 10\sigma^4\right) + 2\kappa\sigma\left(5 - 2\sigma - 6\sigma^2 + 3\sigma^3\right) + 2\kappa(1-\sigma)^2\sigma\left(1 + 5\sigma^2 - 3\sigma^3\right) - (1-\sigma)^3\sigma^2.$$

Clearly, $X < 0$. If $Y < 0$ then we are done. Suppose now that $Y > 0$. Then $AX + Y < 0$ is equivalent to $Y^2 - (AX)^2 < 0$, which in turn is equivalent to

$$\kappa^4 + 4\kappa^3(1-\sigma) + 2\kappa^2\left(2 + 2\sigma - \sigma^2\right) + 4\kappa\sigma\left(2 - 5\sigma + 3\sigma^2\right) + (1-\sigma)^2\sigma(2 - 3\sigma) > 0$$

which is always true.

Item (iv) First note that $H_1^* = \frac{\alpha\left(\kappa^2 - A - \sigma(1-\sigma)\right)(\kappa(\kappa+2) - A + \sigma(\sigma+1))}{4\sigma(\sigma-1-\kappa)(\kappa+\sigma)} = \frac{\left(\kappa^2 - A - \sigma(1-\sigma)\right)}{2(\sigma-1-\kappa)}\lambda_1^*$. We can show that $\frac{\partial H_1^*}{\partial \kappa} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -2\kappa^5 - 6\kappa^4 - \kappa^3\left(6 + 4\sigma - 4\sigma^2\right) - \kappa^2\left(2 + 7\sigma - 6\sigma^2\right) - 2\kappa\sigma\left(2 - 3\sigma + 4\sigma^2 - 3\sigma^3\right) - (1-\sigma)\sigma^2, \text{ and}$$
$$Y = 2\kappa^7 + 10\kappa^6 + 2\kappa^5\left(9 + \sigma - \sigma^2\right) + \kappa^4\left(14 + 15\sigma - 16\sigma^2\right) + \kappa^3\left(4 + 24\sigma - 32\sigma^2 + 16\sigma^3 - 10\sigma^4\right) + 2\kappa^2\sigma\left(5 - 2\sigma - 6\sigma^2 + 3\sigma^3\right) + 2\kappa\sigma^2\left(3 - 4\sigma - 3\sigma^2 + 7\sigma^3 - 3\sigma^4\right) + (1-\sigma)^2\sigma^3.$$

Clearly, $X < 0$ and $Y > 0$. Hence, $AX + Y < 0$ is equivalent to $Y^2 - (AX)^2 < 0$ which simplifies to $-\left(\kappa^4 + 4\kappa^3\sigma - 2\kappa^2\left(1 - 4\sigma + \sigma^2\right) - 4\kappa\sigma\left(1 - 4\sigma + 3\sigma^2\right) + \sigma^2\left(-1 + 4\sigma - 3\sigma^2\right)\right) <$

0. Hence, we have

$$\frac{\partial H_1^*}{\partial \kappa} < 0 \Leftrightarrow \kappa^4 + 4\kappa^3\sigma - 4\kappa\sigma\left(1 - 4\sigma + \sigma^2\right) - \left(\sigma^2 + 4\kappa\right)(1-\sigma)(1-3\sigma) > 0.$$

If $\sigma > \frac{1}{3}$ then we have $\frac{\partial H_1^*}{\partial \kappa} < 0$. Also, if $\kappa > 2$, then $\frac{\partial H_1^*}{\partial \kappa} < 0$. We now focus on $\sigma \in \left(0, \frac{1}{3}\right)$ and $\kappa \in (0, 2)$. Let $g(\kappa) = \kappa^4 + 4\kappa^3\sigma - 4\kappa\sigma(1 - 4\sigma + \sigma^2) - (\sigma^2 + 4\kappa)(1-\sigma)(1-3\sigma)$. Then $g'(\kappa) = 4\kappa^3 + 12\kappa^3\sigma - 4\sigma(1 - 4\sigma + \sigma^2) - 4\kappa(1-\sigma)(1-3\sigma)$. Note that $g(0) < 0$ and $g(2) = 8 + 56\sigma + 23\sigma^2 - 50\sigma^3 - 3\sigma^4 > 0$. Hence, there will be at least one root between 0 and 2 for $g(\kappa) = 0$. However, $g'(\kappa) > 0$ for $\kappa \in (0, 2)$. Therefore, by Rolle's theorem, there will only be one root.

Proposition 5

Item (i) Note that $\lambda_1^*\lambda_2^* = \left(\frac{\kappa(\kappa+2)+\sigma(\sigma+1)-A}{2\sigma(\kappa+\sigma)}\alpha \frac{\kappa(\kappa+2)+\sigma(\sigma-3)+2-A}{2(1-\sigma)(\kappa+1-\sigma)}\alpha\right)$.

With some algebra, we can show that $\frac{\partial}{\partial\sigma}\lambda_1^*\lambda_2^* > 0$ is equivalent to $9\kappa^4+3\kappa^5+\kappa^3(8 + 10\sigma - 10\sigma^2)+\kappa^2(2 + 14\sigma - 14\sigma^2) + \kappa\sigma(4 - \sigma - 6\sigma + 3\sigma^3) + (1-\sigma)^2\sigma^2 > 0$. We can rewrite it as

$$AX + Y < 0$$

where

$$\begin{aligned} X &= 3\kappa^4 + 6\kappa^3 + 2\kappa^2\left(2 - \sigma + \sigma^2\right) + \kappa\left(1 - 2\sigma + 2\sigma^2\right) - (1-\sigma)^2\sigma^2, \text{ and} \\ Y &= -3\kappa^6 - 12\kappa^5 - \kappa^4\left(18 - 5\sigma + 5\sigma^2\right) - 11\kappa^3 - \kappa^2\left(2 + 6\sigma - 5\sigma^2 - 2\sigma^3 + \sigma^4\right) \\ &\quad - \kappa\sigma\left(1 + 3\sigma - 8\sigma^2 + 4\sigma^3\right). \end{aligned}$$

Note that Y is always negative. If $X \leqslant 0$, then we are done. Consider $X > 0$, then $AX + Y < 0$ is equivalent to

$$(AX)^2 - Y^2 < 0$$

which is equivalent to $9\kappa^4+3\kappa^5+\kappa^3(8 + 10\sigma - 10\sigma^2)+\kappa^2(2 + 14\sigma - 14\sigma^2)+\kappa\sigma(4 - \sigma - 6\sigma + 3\sigma^3)+(1-\sigma)^2\sigma^2 > 0$ which is always true.

Item (ii) To show that $\frac{\partial H_1^*}{\partial\sigma} > 0$, we rely on the fact that $\frac{\partial H_2^*}{\partial\sigma} < 0$ and $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial\sigma} < 0$, which are shown below. We can show that $\frac{\partial H_2^*}{\partial\sigma} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -\kappa^6 - 4\kappa^5 - \kappa^4\left(2+8\sigma-5\sigma^2\right) + 2\kappa^3\left(1-7\sigma+5\sigma^2\right)$$
$$+\kappa^2\left(3-16\sigma+24\sigma^2-16\sigma^3+5\sigma^4\right) + \kappa(1-\sigma)^2\left(1-4\sigma+2\sigma^2\right) - (1-\sigma)^4\sigma^2, \text{ and}$$
$$Y = \kappa^8 + 6\kappa^7 + \kappa^6\left(10+7\sigma-4\sigma^2\right) + \kappa^5\left(2+30\sigma-20\sigma^2\right) + \kappa^4\left(-11+58\sigma-62\sigma^2+29\sigma^3-10\sigma^4\right)$$
$$+\kappa^3\left(-9+32\sigma-25\sigma^2+2\sigma^4\right) - \kappa^2(1-\sigma)^2\left(2+\sigma-4\sigma^2-9\sigma^3+4\sigma^4\right) - (1-\sigma)^5\sigma^3.$$

Consider first the case $\kappa \geqslant 1$. We have $X < 0$. Then if $Y < 0$, the inequality $(AX+Y<0)$ is true. Now consider $Y > 0$. Then $AX + Y < 0$ is equivalent to

$$Y^2 - (AX)^2 < 0$$

which is always true.

Now consider $\kappa \in (0,1)$. First, $\sigma > \frac{1}{4}(3+4\kappa) - \frac{1}{4}\sqrt{1+16\kappa+32\kappa^2}$ implies $X < 0$. Then if $Y < 0$, the inequality $(AX+Y<0)$ is true. Now consider $Y > 0$. Then $AX+Y<0$ is equivalent to

$$Y^2 - (AX)^2 < 0.$$

If $\sigma < \frac{1}{4}(3+4\kappa) - \frac{1}{4}\sqrt{1+16\kappa+32\kappa^2}$, $Y < 0$. Then if $X < 0$, the inequality $(AX+Y<0)$ is true. Now consider $X > 0$. Then $AX+Y<0$ is equivalent to

$$(AX)^2 - Y^2 < 0,$$

which can be shown to be true.

Item (iii) Note that

$$\frac{\partial\left(H_1^* + H_2^*\right)}{\partial\sigma} = \frac{\alpha\kappa(1-2\sigma)\left(-\kappa^3-4\kappa^2+\kappa\left(A+\sigma-\sigma^4-4\right)+2(A-(1-\sigma)\sigma)\right)}{2(1-\sigma)^2\sigma^2 A}.$$

Because the denominator is positive, $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial\sigma} > 0$ is equivalent to $-\kappa^3-4\kappa^2+\kappa\left(A+\sigma-\sigma^4-4\right)+2(A-(1-\sigma)\sigma) > 0$. We can rewrite the above as

$$AX + Y > 0$$

where

$$X = -(2+\kappa), \text{ and}$$
$$Y = \kappa^3 + 4\kappa^2 + \kappa\left(4 - \sigma + \sigma^2\right) + 2(1-\sigma).$$

As $X < 0$, and $Y > 0$, then $AX + Y > 0$.

Proposition 6

We can show that

$$\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} = \lambda_1^* - \lambda_2^* + \sigma \frac{\partial \lambda_1^*}{\partial \sigma} + (1-\sigma)\frac{\partial \lambda_2^*}{\partial \sigma} < 0$$

is equivalent to

$$8\kappa\left(\kappa + \kappa^2 + \sigma - \sigma^2\right)^2 > 0,$$

which is always true. As $\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0$ and $\frac{\partial\left(H_1^* + H_2^*\right)}{\partial \sigma} > 0$, we have

$$\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} = \frac{\left(H_1^* + H_2^*\right)\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} - \mathcal{H}^{1,2}\frac{\partial\left(H_1^* + H_2^*\right)}{\partial \sigma}}{\left(H_1^* + H_2^*\right)^2} < 0.$$

Proposition 7

By Proposition 5, $\frac{\partial\left(H_1^* + H_2^*\right)}{\partial \sigma} > 0$. Moreover, $H_A^* = \frac{H_1^* + H_2^*}{1+\kappa q}$ and $H_B^* = \frac{\kappa q\left(H_1^* + H_2^*\right)}{(1+\kappa q)(2+\kappa q)}$, therefore $\frac{\partial H_A^*}{\partial \sigma} > 0$ and $\frac{\partial H_B^*}{\partial \sigma} > 0$.

Proposition 8

We have $H_A^* + H_B^* = \frac{H_1^* + H_2^*}{1+q\kappa} + \frac{\kappa q\left(H_1^* + H_2^*\right)}{(1+\kappa q)(2+\kappa q)} = \left(H_1^* + H_2^*\right)\left(\frac{2}{\kappa q + 2}\right)$ and $\frac{\mathcal{H}^{A,B}}{H_A^* + H_B^*} = \frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*}\frac{2+\kappa q}{2(1+\kappa q)}$. Hence, we have

$$\frac{\partial \mathcal{H}^{A,B}}{\partial \sigma} = \frac{1}{1+\kappa q}\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0,$$

and

$$\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{A,B}}{H_A^* + H_B^*} = \frac{2+\kappa q}{2(1+\kappa q)}\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} < 0.$$